ON DEFORMATION MODELS OF THE THEORY OF PLASTICITY AND COMPLEX MEDIA

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Models of different complex media based on the introduction of a central friction mechanism, are considered. Such plasticity mechanisms allow proposing a foundation for deformation plasticity theories different from those known [1–4]. Also considered is a model of plasticity theory which generalizes the theory of ideal plasticity by describing effects characteristic of the theory of anisotropic hardening without the introduction of "elastic microstresses".

1. In examining the properties of varius continua manifested during quasistatic loading, it is useful to utilize mechanical analogs in a number of cases which permit graphical illustration of some properties of media. When modelling the properties of an elastic medium, ordinarily an elastic spring (Fig. 1a) is considered, while a piston in a viscous fluid [a dashpot] (Fig. 1b) and a body on a frictive surface (Fig. 1c) are considered for the viscous and plastic properties, respectively. The inertial properties of the models are not examined.

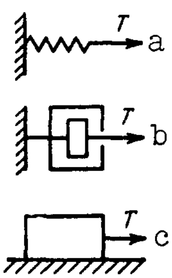

Fig. 1

The models mentioned can be represented somewhat differently, more conveniently for the subsequent generalizations.

Let us imagine that an elastic element is connected by a flexible inextensible filament spanning an ideal block, to an element located on a horizontal plane without friction to which a stress resultant T is applied (Fig 2a). Evidently this mechanical model does not differ, in its properties, from the model pictured in Fig. 1a. Mechanical viscosity and friction models can be introduced analogously (Figs. 2b, c). Evidently, the intrinsic weight of the elements, besides the inertial properties, should not be considered in the models pictured in Fig. 2, such as for example, the weight of the piston of the viscous element, or the weight of the friction element in Figs. 2b, c.

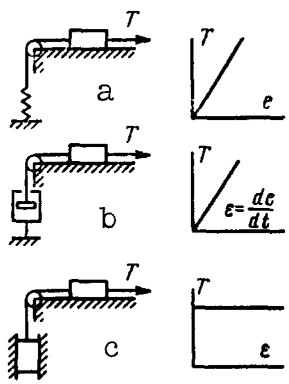

Fig. 2

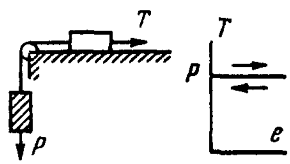

Fig. 3

A scheme with vertically disposed mechanisms can be utilized for the introduction of a model whose element is a load of weight P (Fig. 3). In this case the force-displacement diagram under loading evidently has a form analogous to the model with friction

(Fig. 2c), but the loading and unloading processes are reversible and, in substance, phenomena characteristic for nonlinearly elastic bodies occur. Let us note that utilization of a gravity element does not introduce any principles for the determination of elastic body models: springs with nonlinear characteristics and a variable gravity field (an equivalent state is achieved by changing the mass of the load, for example) result in models with identical mechanical properties.

A scheme with vertically disposed mechanisms permits the introduction of two-dimensional models of plastic, elastoplastic, and other complex media different from those considered earlier, and whose properties can underlie the foundation of deformation theories of these media.

Let us present a two-dimensional model of an elastic body. A stress resultant T with components T_1, T_2 is applied to a flexible inextensible filament at a point A on a horizontal plane. Let Δe, with the components Δe_1, Δe_2, denote the increment in the displacement at the point A.

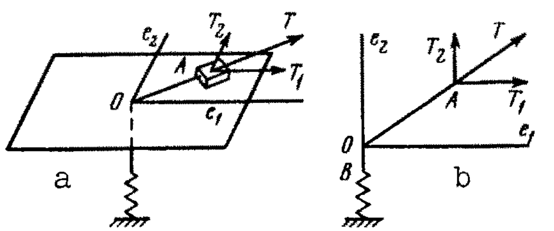

Fig. 4

Let $e = \sqrt{e_1^2+e_2^2}$ be called the intensity of displacement. The filament AB passes through a hole at the origin and connects with the vertically disposed elastic spring at the point B (Fig. 4a). The same model is shown in Fig. 4b schematically. Analogous schemes of two-dimensional models for a viscous, plastic, elastoplastic, plastic hardening and viscoplastic body are presented in Figs. 5a-e, respectively.

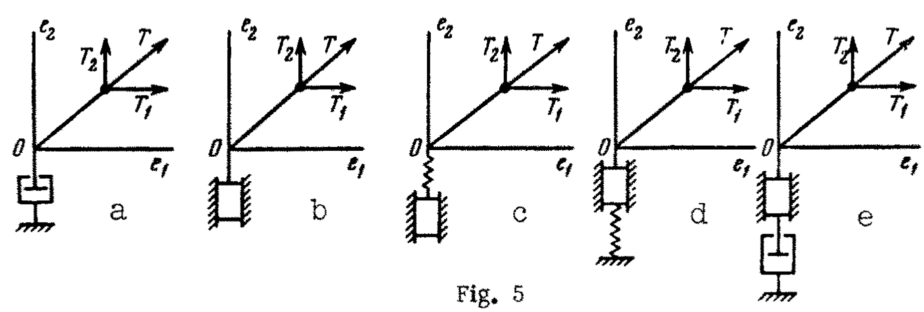

Fig. 5

In all the two-dimensional models shown in Figs. 4 and 5, the element in the horizontal plane is ideally smooth, there are no viscous friction forces between them and the horizontal plane.

Let us form the fundamental equations governing the behavior of the mechanical models pictured in Figs. 4 and 5.

For the elastic model (Fig. 4) we will have

$$\frac{e_1}{T_1} = \frac{e_2}{T_2}, \qquad \sqrt{T_1^2+T_2^2} = c\sqrt{e_1^2+e_2^2} \tag{1.1}$$

where c is the stiffness coefficient of the spring.

For all the subsequent models (Fig. 5), proportionality between the displacements and stress resultants

$$\frac{e_1}{T_1}=\frac{e_2}{T_2} \tag{1.2}$$

will also hold, independently of the nature of the mechanisms.

For the viscous body model (Fig. 5a)

$$\sqrt{T_1^2+T_2^2}=\mu\sqrt{\varepsilon_1^2+\varepsilon_2^2}, \qquad \varepsilon_i=\frac{de_i}{dt} \tag{1.3}$$

holds, where μ is the viscosity factor.

For the model of a plastic body (Fig. 5b)

$$T_1^2+T_2^2=k^2, \qquad (k=\text{const}) \tag{1.4}$$

holds, where k is the limit value of the dry friction force.

For the model of an elastoplastic body (Fig. 5c)

$$T_1^2+T_2^2\leqslant k^2, \qquad \sqrt{T_1^2+T_2^2}=c\sqrt{e_1^2+e_2^2} \tag{1.5}$$

holds.

For the model of a plastic hardening body (Fig. 5d)

$$\sqrt{T_1^2+T_2^2}=k+c\sqrt{e_1^2+e_2^2} \tag{1.6}$$

holds.

Finally, for the model of a viscoplastic body there holds

$$\sqrt{T_1^2+T_2^2}=k+\mu\sqrt{\varepsilon_1^2+\varepsilon_2^2} \tag{1.7}$$

Two-dimensional models for different complex media can be considered analogously.

2. Let us examine the plastic body model (Fig. 5b) in more detail. As has already been established, the fundamental relationships are (1.2), (1.4). In the general case the displacements e_1, e_2 are not residual. The magnitude of the intensity of displacement $e=(e_1^2+e_2^2)^{1/2}$ is a measure of the residual strain.

Displacement of an element under a constant intensity of displacement (neutral loading) along a circle AA' (Fig. 6) will occur without the stress resultant performing work on the displacements (this circumstance is characteristic for all the two-dimensional models introduced).

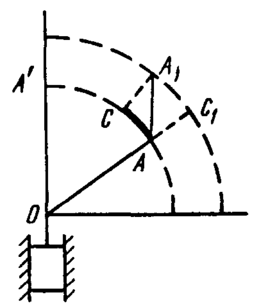

Fig. 6

The neutral loading process is completely reversible.

Let us examine the loading resulting in displacements from the point A to the point A_1 (Fig. 6).

The displacements AA_1 are composed of the vector sum of the displacements $AC_1(\Delta e^n_1,\ \Delta e^n_2)$ normal, and $AC(\Delta e^\tau_1,\ \Delta e^\tau_2)$ tangent to the circle AA'.

Displacements of the friction (plasticity) element evidently cause just increments in the normal displacements, and in this sense the plastic deformation law

$$\frac{\Delta e_1^n}{T_1}=\frac{\Delta e_2^n}{T_2}, \qquad T_1^2+T_2^2=k^2 \tag{2.1}$$

is valid, where Δe^n_1, Δe^n_2 are the normal increments in the displacements causing the displacement of the plasticity element.

It should however be kept in mind that increments in the normal displacements causing a residual displacement of the friction element of the magnitude $[(\Delta e^n_1)^2+(\Delta e_2^n)^2]^{1/2}$ will not themselves be residual: they can be reversibly altered by a neutral loading.

Unloading can be defined as follows. It can be assumed that the flexible filament in the model is capable of transmitting the stress resultant in the opposite direction. Then the model under consideration will behave as a rigidly plastic body. Unloading holds when stress resultants assure an increment in the intensity of displacement Δe of opposite sign, whereupon the relationships (1.2), (1.4) are satisfied.

Let us assume that the element has received a displacement of intensity e, and then the loadings are removed. An element in a horizontal plane without stress resultants can occupy any position under a constant intensity e (any position on the circle AA', Fig. 6).

To some extent the behavior of an ideal fluid, an isolated element of which can change shape arbitrarily, is the analog of such behavior.

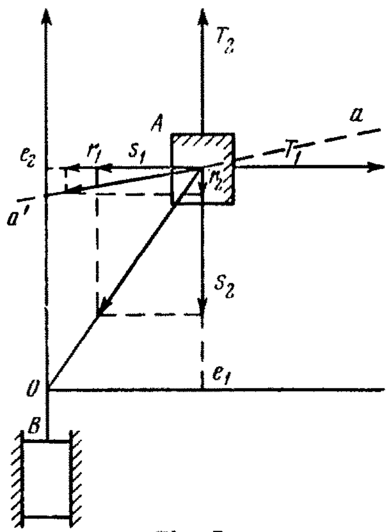

Fig. 7

In passing, let us note that a body on a smooth horizontal plane can be a two-dimensional model illustrating the properties of an ideal fluid.

Strain without a change in the potential loading under a constant stress intensity also exists for an incompressible elastic body.

3. Let us examine the relationships of the deformation theory of ideal plasticity. Upon utilization of dynamic analogies, stress deviators are set in correspondence with the stress resultants, and strain deviators with the displacements. The dependence between the first invariants of the stress and strain tensors is formulated independently.

Let us set the plasticity condition

$$\sigma_{ij}'\sigma_{ij}' = k^2, \qquad k = \text{const} \tag{3.1}$$

in correspondence with condition (1.4).

Here and henceforth, the prime denotes the deviator components.

The relationships

$$e_{ij}' = \lambda\sigma_{ij}' \tag{3.2}$$

correspond to condition (1.2).

The quantity λ is defined according to (1.3), (1.4), from which $\lambda = (1/k)(e'_{ij}e_{ij}')^{1/2}$.

The quantity $e_u' = (e_{ij}' \; e_{ij}')^{1/2}$ is considered a measure of the plastic deformation. The plastic deformation is invariant if e_u' is constant, $de_u' = 0$.

The relationship of the deformation theory of a hardening plastic body agrees completely with the relationships of the theory of small elastoplastic deformations

$$e_{ij}' = \frac{e_u'}{\sigma_u'}\sigma_{ij}', \qquad \sigma_u' = \Phi(e_u'), \qquad \sigma_u' = (\sigma_{ij}'\sigma_{ij}')^{1/2} \tag{3.3}$$

Under this interpretation the plastic deformation is connected with the quantity de_u'. There is no plastic deformation for $de_u' = 0$.

Let us note that the relationships of the deformation theory of a viscoplastic body and similar models connected with the models introduced can be written down completely analogously.

4. Let us discuss one model of a plastic body. Let us consider the following two-dimensional model (Fig. 7). A dry friction element A is placed on a horizontal plane, and is in turn connected by a tension AB to a dry friction element B. Under the effect of the external stress resultants T_1, T_2 the element is displaced in the direction of the line aa'.

The reaction of the dry friction element A has the components r_1, r_2, and their resultant is directed opposite to the motion of the element, i. e. along the line aa_1. The stress resultants transmitted by the dry friction element are denoted by s_1, s_2. In the ultimate state we evidently have

$$T_1 = s_1 + r_1, \quad T_2 = s_2 + r_2, \quad s_1^2 + s_2^2 = k_1^2, \quad r_1^2 + r_2^2 = k_2^2$$

$$\frac{e_1}{s_1} = \frac{e_2}{s_2}, \quad \frac{de_1}{r_1} = \frac{de_2}{r_2} \quad (k_1, k_2 = \mathrm{const}) \tag{3.4}$$

From (3.4) we obtain

$$T_1^2 + T_2^2 = k_1^2 + k_2^2 + 2(s_1r_1 + s_2r_2) \tag{3.5}$$

Let us note that

$$s_i = \frac{k_1 e_i}{\sqrt{e_1^2 + e_2^2}}, \qquad r_i = \frac{k_2 de_i}{\sqrt{de_1^2 + de_2^2}} \qquad (i = 1, 2) \tag{3.6}$$

Then the relationship (3.5) is easily converted to

$$T_1^2 + T_2^2 = k_1^2 + k_2^2 + 2k_1k_2\left(\frac{e_1de_1 + e_2de_2}{\sqrt{e_1^2 + e_2^2}\sqrt{de_1^2 + de_2^2}}\right) \tag{3.7}$$

From (3.4) and (3.6) we also obtain

$$\frac{de_1}{T_1 - k_1e_1/\sqrt{e_1^2 + e_2^2}} = \frac{de_2}{T_2 - k_2e_2/\sqrt{e_1^2 + e_2^2}} \tag{3.8}$$

Let $dD = T_1de_1 + T_2de_2$. From (3.4) we obtain

$$dD = k_1d\left(\sqrt{e_1^2 + c_2^2}\right) + k_2\sqrt{(de_1)^2 + (de_2)^2} \tag{3.9}$$

It follows from (3.7), (3.8) that the maximal external stress resultant occurs when the directions e_i, de_i coincide , in this case

$$T_1^2 + T_2^2 = (k_1^2 + k_2^2) \tag{3.10}$$

In all the other cases the external stress resultant is less than (3.10). It follows from (3.9) that, with the exception of the case of coincidence in the directions of e_i and de_i, the directions de_i and T_i do not coincide, i. e. phenomena occur of the type of acquired anisotropy described by the theory of translational hardening, for example. However, in this case there are no "elastic microstresses"; all the mechanisms underlying the construction of the model are irreversible.

Let us write down analogous relationships of plasticity theory. The expressions

$$\sigma_{ij}' = s_{ij}' + r_{ij}', \qquad s_{ij}'s_{ij}' = k_1^2, \qquad f(r_{ij}') = k_2^2, \qquad k_1, k_2 = \mathrm{const} \tag{3.11}$$

$$e_{ij}' = \lambda s_{ij}', \qquad de_{ij}' = d\mu\frac{\partial f}{\partial r_{ij}}$$

correspond to conditions (3.4).

In the simplest case

$$f(r_{ij}') = r_{ij}'r_{ij}' = k_2^2$$

Then

$$\sigma_{ij}'\sigma_{ij}' = k_1^2 + k_2^2 + \frac{2k_1k_2e_{ij}'de_{ij}'}{(e_{ij}'e_{ij}')^{1/2}(de_{ij}'de_{ij}')^{1/2}}$$

$$de_{ij}' = d\mu\left[\sigma_{ij}' - \frac{k_1e_{ij}'}{(e_{ij}'e_{ij}')^{1/2}}\right] \tag{3.12}$$

The maximal quantity in the proposed theory is $\sigma_u = k_1 + k_2$, and is generally dependent on the cosine of the angle between the vectors e_{ij}' and de_{ij}'. The relationships of classical ideal plasticity theory occur from (3.11) in the particular case with $k_1 = 0$, $k_2 \neq 0$, and of deformation ideal plasticity theory for $k_1 \neq 0$, $k_2 = 0$.

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DYNAMICS OF FREE SYSTEMS OF MATERIAL POINTS WITH ELASTIC CONSTRAINTS

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A method is indicated for constructing the equations of motion of free systems of material points connected to each other by inertialess elastic constraints. The system configuration is arbitrary, and any arbitrarily time-dependent external forces are applied to the material points.

1. Rigid system. Let us assume that there are N material points in the system. Let m_i denote the mass of the ith point, M the sum of all the masses so that

$$M = \Sigma m_i \tag{1.1}$$

Here, as everywhere below, the symbol Σ denotes summation over all material points of the system, i. e. over i between 1 and N.

We call a system for which the deformations of the elastic constraints are zero – a rigid system. If the constraints are absolutely rigid, the rigid system is substantially an absolutely rigid body.

Let us refer the rigid system to fixed Cartesian coordinates with the unit vectors $\mathbf{e}_j^\circ$ $(j = 1,2,3)$, and also to moving coordinates with $\mathbf{e}_j$ $(j = 1,2,3)$ directed along the principal central axes of inertia of the rigid system. If ρ_i°, ρ_c° are radius-vectors of the ith point and the center of mass of the rigid system relative to the origin of the fixed coordinates, and

$$\rho_i = x_i^1\mathbf{e}_1 + x_i^2\mathbf{e}_2 + x_i^3\mathbf{e}_3 \tag{1.2}$$

is the radius-vector of the ith point of a rigid system relative to its center of mass, then

$$\rho_i^\circ = \rho_c^\circ + \rho_i \tag{1.3}$$

By the definition of the center of mass